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Some examples of static equivalency in space using descriptive geometry and Grassmann algebra

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Abstract

This paper presents some examples of three-dimensional static equivalency using descriptive geometry and Grassmann algebra. In many static problems, it is convenient to replace existing force system with another, usually simpler, statically equivalent force system. Procedures for replacing two forces with two other forces which fulfil certain conditions (one of the forces on a given line or a force through a given point and a force in a given plane, including special cases with a point or a plane at infinity), thus providing unique replacement of a system of forces with two forces, and the procedure for replacing a single force with three components on three skew lines are described in the main part of the paper.

Keywords: 3D graphic statics, static equivalency, descriptive geometry, Grassmann algebra, Plücker coordinates, line geometry

1. Introduction

The revival of the interest in graphic statics and its extensions into the third dimension is motivated by the understanding that it is a suitable and valuable tool in the design of free-form structures, but also in engineering education. It is a broad field of research which can be approached in various ways, as witnessed, for example, by numerous articles presented on sessions dedicated to graphic statics on former IASS conferences and by the recent volume of *International Journal of Space Structures* guest editors of which were P. Block, C. Fivet and T. Van Mele [1].

When a system of external forces is acting on a body, it is easier to understand their overall effect on the body if they are replaced by a simpler system having the same external effect. Two systems of forces are statically equivalent if their contribution to the conditions of static equilibrium is the same. It is well known that a general system of forces in space can be replaced by an equivalent system containing a resultant force and a resultant couple or by an equivalent system containing two forces called conjugate forces (and lines along which they act are called conjugate lines). Geometric procedures for replacing a system of three skew (i.e. mutually nonintersecting) forces with two skew forces was described by Schrems and Kotnik [2] and by D'Acunto et al. [3]. Fourth force can be added to these two forces and the new system of three forces can again be replaced with two forces. Thus, a system of skew forces can be reduced to two forces. However, as noted by D'Acunto et al. [3], this reduction is not unique – there is multiple infinity of possible solutions. Additional conditions which ascertain unique reduction are reviewed in the third section of this article. Besides, an interesting example of a resolution of a single force into a sum of three skew components is presented.

Graphical procedures for replacing given system of forces with some other system are carried out using operations of descriptive geometry. We concentrate on operations of descriptive geometry because it emphasizes "visual thinking" which is, from the educational standpoint at least, essential for graphical statics. Using Grassmann algebra, procedures of descriptive geometry can be readily translated into algebraic expression (Grassmann [4], Whitehead [5]) and then into computer program code. Grassmann algebra and Plücker coordinates of a line are very briefly introduced in the next section.

2. Grassmann algebra

A line can be specified as a span of two points and a plane as a span of three points or a point and a line. A line can also be specified as an intersection of two planes, while a point can be specified as an intersection of three planes or a line and a plane. If they intersect, two lines specify a plane and a point. In Grassman algebra spans and intersections are defined as products, progressive and regressive, of points, lines and planes (Grassmann [4], Whitehead [5]).

Using homogeneous coordinates, points are written in the form $X = (x_0, x_1, x_2, x_3)$. If $x_0 \neq 0$, point is Euclidean, and if $x_0 = 0$, point is at infinity. As X and aX, with $a \neq 0$, denote the same point, Euclidean point can be written in the form $(1, x, y, z)$, where x, y, z are its Cartesian coordinates. A plane defined by equation $u_0x_0 + ... + u_3x_3 = 0$ can be expressed in coordinate form $\upsilon = (u_0, u_1, u_2, u_3)$, where (u_1, u_2, u_3) is the normal vector of the plane. A plane at infinity has coordinates $\upsilon = (u_0, 0, 0, 0)$, with $u_0 \neq 0$.

Spans and intersections of points, lines, and planes can be computed using progressive and regressive products. In particular, a line, as a span of points X and Y, can be expressed in the form $l = [X\ Y] = [(x_0, x_1, x_2, x_3)\ (y_0, y_1, y_2, y_3)] = (l_{01}, l_{02}, l_{03}, l_{23}, l_{31}, l_{12})$, where $l_{ij} = x_iy_j - x_jy_i$ are homogeneous coordinates of a line, called Plücker coordinates (Plücker [6], Pottmann and Wallner [7]). Analogously, as an intersection of two planes, a line can be expressed in the form $l^* = [\alpha\ \beta] = (l_{01}^*, l_{02}^*, l_{03}^*, l_{23}^*, l_{31}^*, l_{12}^*)$, where l_{ij}^* are called axis coordinates. It can be shown that $(l_{01}, l_{02}, l_{03}, l_{23}, l_{31}, l_{12}) = (l_{23}^*, l_{31}^*, l_{12}^*, l_{01}^*, l_{02}^*, l_{03}^*)$.

According to the principle of transmissibility, a force can be shifted along its line of action. Therefore, the force is considered as a line-bound vector with Plücker coordinates $\boldsymbol{F} = (f_{01}, f_{02}, f_{03}, f_{23}, f_{31}, f_{12})$. The first three coordinates represent vector $\boldsymbol{f}$ of the force $\boldsymbol{F}$, and the second three represent moment vector $\boldsymbol{m}$ of $\boldsymbol{f}$ about the origin of the coordinate system. It should be noted that force coordinates f_{ij} are not homogeneous, because forces have definite intensities. Also, it can be shown that $\boldsymbol{f} \cdot \boldsymbol{m} = 0$, where $\cdot$ denotes dot product (the same holds for line coordinates).

3. Examples of the static equivalency

In this section we describe some examples of static equivalency using descriptive geometry. To perform and visualize described procedures, we are developing a computer program based on algebraic translations of descriptive geometry operations. The code is written in GhPython [8] (Python interpreter component and plug-in for Grasshopper [9]) and the procedures are visualized in Rhinoceros [10].

Most of our geometric constructions can be regarded as partial three-dimensional extensions of construction of funicular polygon (in genuine Varignon's meaning of the words – rope stretched by applied forces), based on two principles: (1) single force can be resolved into two components along two given lines if and only if its line of action and the two given lines are concurrent and coplanar, and (2) when constructing funicular polygon, each of two given forces is resolved into two components in such a way that one component of the first force and one component of the second force cancel each other (these two components lie on the same line and are equal in magnitude, but opposite in sense).

Schrems and Kotnik [2] and D'Acunto et al. [3] have shown how a system of forces can be reduced to two forces. As this reduction is not unique, two replacements of two forces with two other forces, restricted in some way, are described in subsections 3.2 and 3.3.

3.1. Replacing single force with a force acting at a given point and a force lying in a given plane

As a preparation for the procedure described in subsection 3.2 we give geometric "translation" of the proof that "[a]ny force can be resolved into a sum of two forces, of which one passes through a given point and one lies in a given plane, which does not contain the point", as given by Whitehead [5].

A plane σ is defined by the given point A and the line of action s of the given force $\boldsymbol{S}$ (in algebraic terms: progressive product $\sigma = [A\ s]$). First component of the force $\boldsymbol{S}$ acts along the line r, which is the intersection line of the plane σ and a given plane β (regressive product: $r = [\sigma\ \beta]$). Second component acts along the connecting line p of the point A and the intersection point P of the line s and the plane β (regressive and then progressive product: $P = [s\ \beta]$, $p = [A\ P]$). Previous steps were performed in the form diagram and following steps will be performed in the force diagram. From arbitrarily chosen point O vector $\boldsymbol{s}$ of the force $\boldsymbol{S}$ is drawn; head of $\boldsymbol{s}$ is the point B ($B = O + \boldsymbol{s}$). Lines r' and p' are drawn through O and B parallel to the lines r and p ($r' = [O\ \boldsymbol{r}]$, $p' = [B\ \boldsymbol{p}]$, where $\boldsymbol{r}$ and $\boldsymbol{p}$ are some vectors on lines r and p). Lines r' and p' intersect in the point C ($C = [r'\ p']$). Vectors $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ of force components $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ on lines r and p are $\boldsymbol{s}_1 = C - O$ and $\boldsymbol{s}_2 = B - C$.

3.2. Replacing two forces with a force acting at a given point and a force lying in a given plane

Whitehead [5] also proved that "any system [of forces] can always be represented by two forces of which one lies in a given plane, and one passes through a given point not lying in the plane".

Using the procedure described in previous subsection, each of two given forces can be replaced with a force acting at the given point and a force lying in the given plane, and then a resultant force acting at a given point and a resultant force acting in the given plane can be found.

We will describe another procedure, resembling funicular polygon construction. Let $\boldsymbol{S_1}$ and $\boldsymbol{S_2}$ be given forces acting on lines s_1 and s_2 (figure 1.a; in this and following figures: left form diagram, right force diagram). We define the plane σ_1 by the given point A and the line s_1, and the plane σ_2 by the line s_2 and some point A_1 arbitrarily chosen on the line s_1. The line s_{12} is the intersection of planes σ_1 and σ_2. The component $\boldsymbol{S_{12}}$ of the force $\boldsymbol{S_1}$ and the component $\boldsymbol{S_{21}}$ of the force $\boldsymbol{S_2}$ act along the same line s_{12} and cancel each other. Second component $\boldsymbol{S_{11}}$ of the force $\boldsymbol{S_1}$ acts along the line p_1 connecting the points A and A_1. Now, we can resolve the force $\boldsymbol{S_2}$ in the plane σ_2. In that way given forces $\boldsymbol{S_1}$ and $\boldsymbol{S_2}$ are replaced with the forces $\boldsymbol{S_{11}}$ and $\boldsymbol{S_{22}}$ (figure 1.b).

The line of action of the force $\boldsymbol{S_{22}}$, line s_{22}, intersects the given plane β in the point B. The line p_2 is the join (or span) of the points A and B (figure 1.c). We define the plane σ_{22} by the lines s_{22} and p_2. Planes σ_{22} and β intersect in the line p_3. Now, we resolve the force $\boldsymbol{S_{22}}$ into two components $\boldsymbol{S_{22,1}}$ and $\boldsymbol{S_{22,2}}$ along the lines p_2 and p_3 (figure 1.d), and the force $\boldsymbol{S_{11}}$ into components $\boldsymbol{S_{11,1}}$ and $\boldsymbol{S_{11,2}} = -\boldsymbol{S_{22,1}}$ (figure 1.e). At the end of the procedure, only the components $\boldsymbol{S_{11,1}} = \boldsymbol{R_1}$, a force which acts at the given point A, and $\boldsymbol{S_{22,2}} = \boldsymbol{R_2}$, a force which lies in the given plane β (figure 1.f), remain.

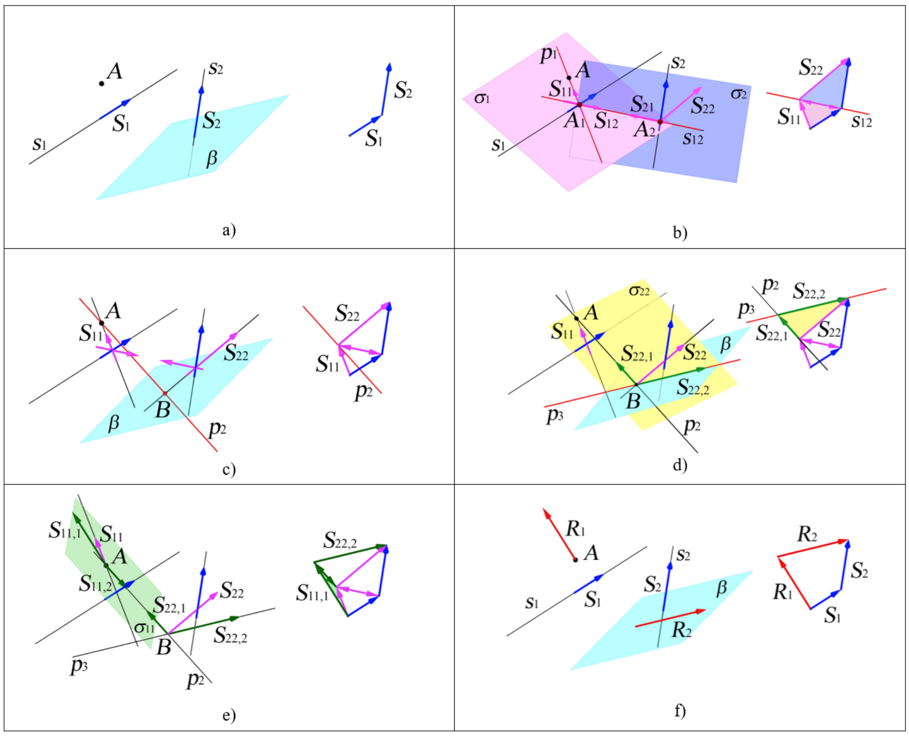

Figure 1: Replacing two forces with a force acting at a given point and a force lying in the given plane

Furthermore, the force $\boldsymbol{R_1}$ can be resolved into three components acting at the point A and the force $\boldsymbol{R_2}$ can be resolved into three nonconcurrent components acting in the plane β. In particular, the point A can be one of the vertices of a tetrahedron, whose opposite face lies in the plane β.

We will rename forces $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ into $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ (figure 2.a). Point A is one vertex of the tetrahedron. Force $\boldsymbol{S}_1$ will be resolved into three components acting along the given lines s_{11}, s_{12} and s_{13}, which are the edges of the tetrahedron concurrent with A. The other three vertices are the intersections T_1, T_2 and T_3 of the lines s_{11}, s_{12} and s_{13} with the plane β. Force $\boldsymbol{S_2}$ will be resolved into three components acting along lines s_{21}, s_{22} and s_{23}, which are the edges of the tetrahedron in the plane β (figure 2.d).

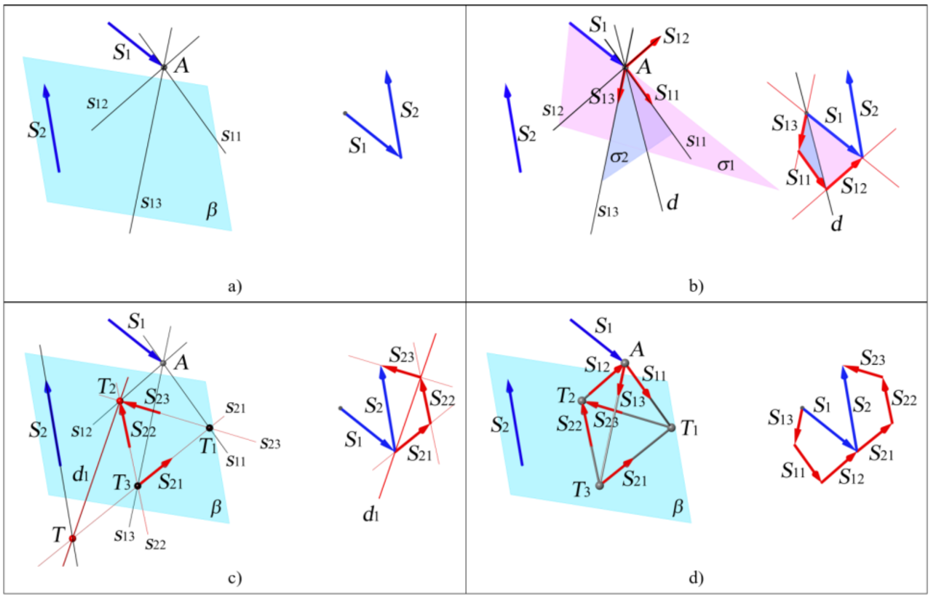

Figure 2: Replacing each of two forces with three forces

The procedure for replacing a single force with three forces acting at the same point (figure 2.b) is described by Jasienski et al. [11] and by Saliklis and Gallion [12]. (The procedure using descriptive geometry is described by Föppl [13].) To resolve the given force $\boldsymbol{S_2}$ in the plane β into three components acting on nonconcurrent lines s_{21}, s_{22} and s_{23} we use the well-known Culmann's method (Culmann [14]) (figure 2.c). Reversion of obtained six forces gives equilibrating forces; their lines of action can be the axes of bars with spherical joints which are supports of some structure.

3.2.1. First special case: A is a point at infinity

Let A be the point at infinity (figure 3.a). Representation of a point at infinity using homogeneous coordinates is $A = (0, a_1, a_2, a_3)$. Plücker coordinates of the line a through the point A and the origin are $a = (a_1, a_2, a_3, 0, 0, 0)$. As all lines containing the point A are parallel to the line a, given forces $\boldsymbol{S_1}$ and $\boldsymbol{S_2}$ will be replaced with a force acting along a line parallel to the line a and a force acting in the given plane β.

We begin with the plane σ_1 in which the force $\boldsymbol{S_1}$ is being resolved into components $\boldsymbol{S_{11}}$ and $\boldsymbol{S_{12}}$. This plane is defined by the line s_1 and the line p_1 which passes through a chosen point A_1 on the line s_1 and is parallel to the line a. Therefore, the plane σ_1 is also parallel to the line a. A plane σ_2 is defined as in the general case (figure 3.b).

Line p_2 is the join of the given point A at infinity and the point B. Therefore, the line p_2 is parallel to the lines a and p_1 (figure 3.c). As before, the force $\boldsymbol{S_{22}}$ is resolved into a component $\boldsymbol{S_{22,1}}$ acting along the line p_2 and a component $\boldsymbol{S_{22,2}}$ acting along the intersection p_3 of the planes β and a plane σ_{22} (figure 3.d).

Now, we resolve the force $\boldsymbol{S_{11}}$ in the plane σ_{11} which is defined by parallel lines p_1 and p_2. One of the components, $\boldsymbol{S_{11,2}} = -\boldsymbol{S_{22,1}}$, acts along the line p_2. As the line p_1 is the line of action of the force $\boldsymbol{S_{11}}$, force component $\boldsymbol{S_{11,1}}$ acts along a line parallel to the lines p_1 and p_2 (figure 3.e). Location and magnitude of the component $\boldsymbol{S}_{11,1}$ can be obtained using the well-known construction of a planar funicular polygon (in the plane σ_{11}).

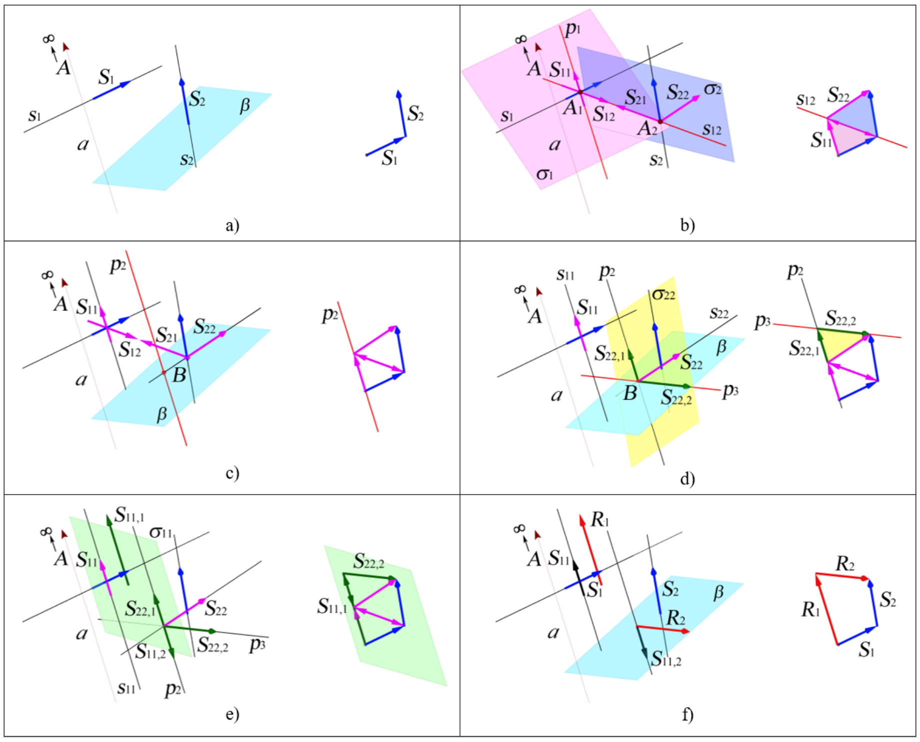

Figure 3: Replacing two forces with a force acting at a point at infinity and a force lying in a given plane

Line of action of the force $\boldsymbol{S}_{11,1} = \boldsymbol{R}_1$ is parallel to the line p_1, and therefore to the line a. The second remaining component $\boldsymbol{S}_{22,2} = \boldsymbol{R}_2$ lies in the given plane β (figure 3.f).

3.2.2. Second special case: β is a plane at infinity

Using homogeneous coordinates, a plane at infinity can be expressed in the form $\beta = (\beta_0, 0, 0, 0)$.

First steps of the procedure (a replacement of given forces $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ with forces $\boldsymbol{S}_{11}$ and $\boldsymbol{S}_{22}$) are the same as in the general case (figure 4.b).

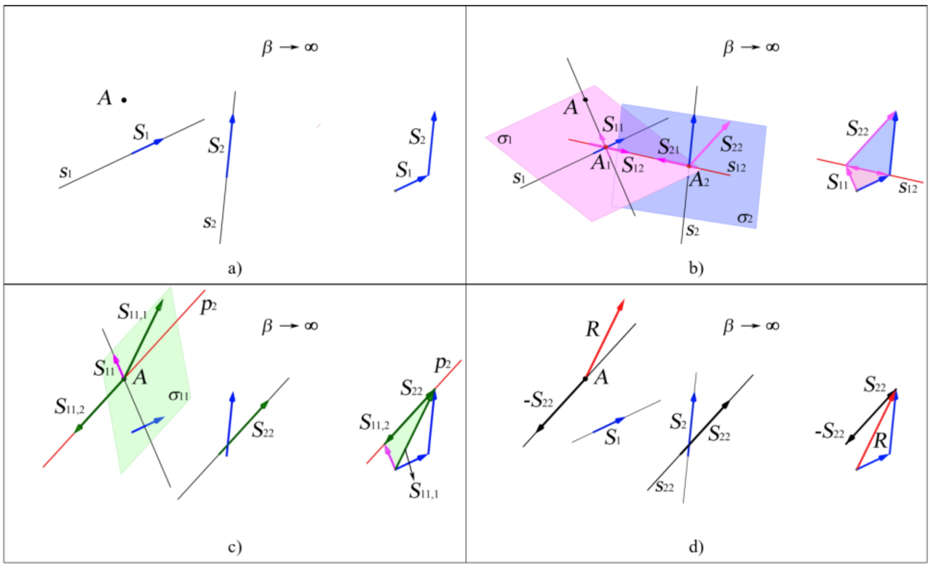

Figure 4: Replacing two forces with a force acting at a given point and a force lying in a plane at infinity

Point *B*, the intersection of the line of action s_{22} of the force $\boldsymbol{S}_{22}$ and the given plane at infinity β, is the ideal point of the line s_{22}. Line p_2, the join of the points *A* and *B*, along which a component $\boldsymbol{S}_{22,1}$ acts, is the line parallel to the line s_{22} (figure 4.c). Planes σ_{22} and the given plane at infinity β intersect in a line at infinity p_3, hence a component $\boldsymbol{S}_{22,2}$ of the force $\boldsymbol{S}_{22}$ acts at infinity. In fact, force $\boldsymbol{S}_{22,2}$ shrinks to a point (it can be said that it is infinitesimal): lines p_2 and s_{22} are parallel and therefore intersect plane at infinity in the same point. Obviously, this force cannot be shown in a figure. However, from the expression $\boldsymbol{S}_{22} = \boldsymbol{S}_{22,1} + \boldsymbol{S}_{22,2}$ follows $\boldsymbol{S}_{22,2} = \boldsymbol{S}_{22} - \boldsymbol{S}_{22,1}$. The forces $\boldsymbol{S}_{22}$ and $\boldsymbol{S}_{22,1}$, equal in magnitudes, opposite in sense and acting along parallel lines, form a force couple. Therefore, an infinitesimal force acting on the line at infinity can be represented by a force couple.

If, as in the general case, $\boldsymbol{S}_{11,2} = -\boldsymbol{S}_{22,1}$, we can in this special case replace given forces $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ with the force $\boldsymbol{R}_1 = \boldsymbol{S}_{11,1}$ and the force couple ($\boldsymbol{S}_{11,2}$, $\boldsymbol{S}_{22}$) (figures 4.c and 4.d).

3.3. Replacing two forces with two forces of which one lies on a given line

In general, a line has one and only one conjugate line with respect to a given system of forces (Möbius [15], Whitehead [5]). Therefore, only one line of the conjugate pair can be chosen arbitrarily.

The line s_0 is a given line and lines s_1 and s_2 are lines of action of the given forces $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ (figure 5.a). We chose points A_1 and A_2 on the lines s_1 and s_2 and connect them with the line s_{12}. The plane σ_0 is defined by the given line s_0 and the point A_1, the plane σ_1 by the lines s_1 and s_{12}, and the plane σ_2 by the lines s_{12} and s_2. Planes σ_0 and σ_1 intersect in the line s_{01} (figure 5.b).

We resolve the given force $\boldsymbol{S}_1$ into components $\boldsymbol{S}_{11}$ and $\boldsymbol{S}_{12}$ acting along the lines s_0 and s_{12}. If, as before, $\boldsymbol{S}_{21} = -\boldsymbol{S}_{12}$, the component $\boldsymbol{S}_{22}$ of the force $\boldsymbol{S}_2$ can be obtained (in the plane σ_2). In this way the given forces $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are replaced with forces $\boldsymbol{S}_{11}$ and $\boldsymbol{S}_{22}$ (figure 5.c).

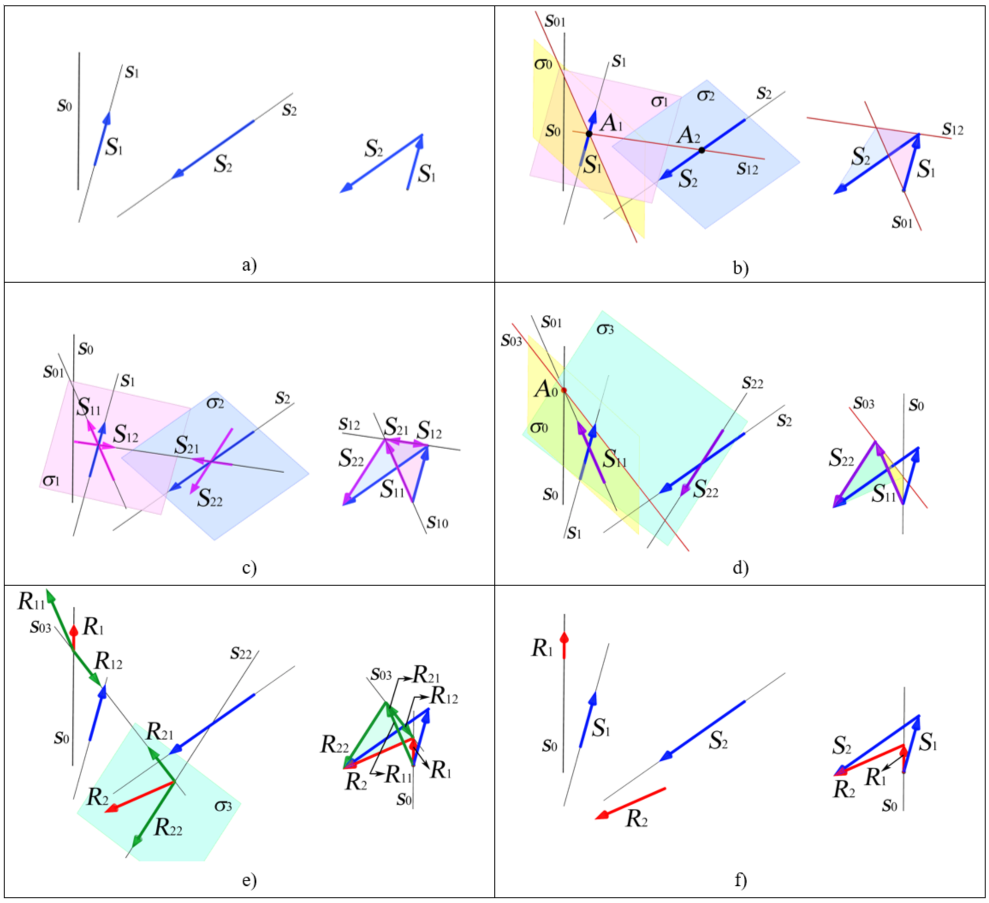

Figure 5: Replacing two forces with a force R_1 lying on a given line s_0 and a force R_2

The line s_{01}, along which the force $\boldsymbol{S}_{11}$ acts, intersects the given line s_0 at the point A_0. The plane σ_3 is the join of the point A_0 and the line s_{22}, along which the force $\boldsymbol{S}_{22}$ acts. Planes σ_0 and σ_3 intersect in the line s_{03} (figure 5.d).

The lines s_0, s_{01} and s_{03} are coplanar and lie in the plane σ_0. Therefore, the force $\boldsymbol{R}_1$, acting along the line s_0, is determined by the conditions that one of its components is $\boldsymbol{R}_{11} = \boldsymbol{S}_{11}$ (acting along the line s_{01}) and the other component is $\boldsymbol{R}_{12}$ acting along the line s_{03}. The force $\boldsymbol{R}_2$ is a resultant force of the forces $\boldsymbol{R}_{21} = -\boldsymbol{R}_{12}$ and $\boldsymbol{S}_{22}$ (figures 5.e and 5.f).

3.4. Replacing a single force with three forces acting along generators of a regulus

As shown in figures 2.b) and 2.c), a force can be resolved into a sum of three concurrent noncoplanar forces or into a sum of three coplanar nonconcurrent forces. A force can also be resolved into a sum of three skew forces if they act along the generators of the same system of a hyperboloid (figure 6.c) (Möbius [15]).

Let the force $\boldsymbol{S}$ and three lines s_1, s_2 and s_3 be given (figure 6.a left). Since all lines are known, the procedure is carried out in the force diagram (figure 6.a right). First, at the tail of the force $\boldsymbol{S}$ we place line s_1' parallel to the line s_1 in the form diagram, and at its head we place line s_3' parallel to the line s_3 in the form diagram.

Through each point in space there is one and only one transversal of two skew lines. The same is the case with points at infinity, which means that a transversal of two skew lines, parallel to the third line, can be found. Therefore, we can close a three-dimensional force polygon in the force diagram by the line s_2' parallel to the line s_2 in the form diagram (figure 6.a right). Transversal of lines s_1' and s_3', parallel to s_2, can be constructed as the intersection of two planes, both parallel to s_2, one containing s_1' and the other containing s_3' (figure 6.a right). The forces $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ and $\boldsymbol{S}_3$, which act along the lines s_1, s_2 and s_3 in the form diagram, are determined by the edges of the obtained closed force polygon in the force diagram (figure 6.b).

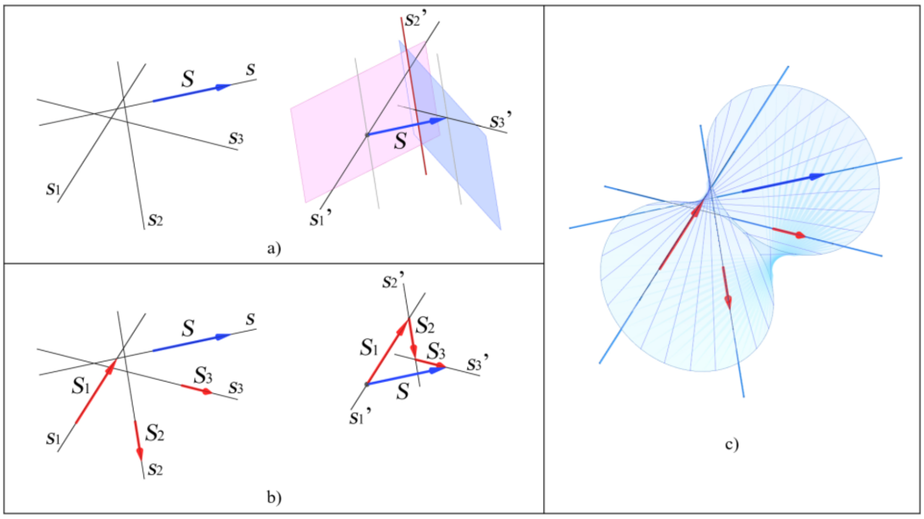

Figure 6: Replacing a single force with three forces acting along generators of a regulus

4. Conclusion

We presented some examples of three-dimensional static equivalency using descriptive geometry and Grassmann algebra. Descriptive geometry can be thought of as "visual language", while Grassmann algebra enables translation of operations of descriptive geometry (even with elements at infinity) into algebraic expression and thereafter into program code.

We described two procedures for replacing two forces with two other forces which fulfil certain conditions, thus providing unique replacement of a system of forces with two forces. One case of further resolution of these forces to fulfil specified boundary conditions was also described: three reactions act on three given nonconcurrent lines in a plane and three reactions act on three given noncoplanar lines through a point and not lying in a plane; point can be at infinity. Future work will address the analysis and possible classification of other dispositions of lines on which reactions act. It is well known that externally statically determinate structure requires six support conditions, for example six bars with spherical joints whose axes are independent lines. In some dispositions of axes reactions can be determined by "direct" geometrical procedure, as in the described case or in the case with three axes through the first point, two axes through the second point and one axis through the third point (D'Acunto et al. [3], Saliklis and Gallion [12]). Some dispositions, on the other hand, require a two-step procedure and some, presumably, even a multi-step procedure; for example, reactions on three pairs of axes through three points can be determined by Henneberg's method of the substitution of bars (Henneberg [16]).

Acknowledgements

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