Ramanujan J
https://doi.org/10.1007/s11139-017-9983-5

Eta-quotients and embeddings of $X_0(N)$ in the projective plane

Iva Kodrnja[1]

Received: 5 October 2016 / Accepted: 29 December 2017

Abstract In this paper, we find projective plane models of the modular curves $X_0(N)$ by constructing maps from $X_0(N)$ to the projective plane using modular forms. We use eta-quotients of weight 12. We find those eta-quotients in $M_{12}(\Gamma_0(N))$ which have maximal order of vanishing at infinity.

Keywords Modular curves · Eta-quotients · Modular forms

Mathematics Subject Classification 11F20 · 11F11

1 Introduction

Let $\Gamma_0(N)$ be the congruence subgroup

$$\Gamma_0(N) = \left\{ \gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}.$$

This group acts on the extended complex upper half plane $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$, with $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$, by linear fractional transformations

$$\gamma.z = \frac{az+b}{cz+d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

The author acknowledges Croatian Science Foundation Grant No. 9364.

✉ Iva Kodrnja
ikodrnja@grad.hr

1 Faculty of Civil Engineering, Kačićeva 26, Zagreb, Croatia

Published online: 23 February 2018

The quotient space of this group action is called a modular curve, and we denote it by $X_0(N)$ i.e.,

$$X_0(N) = \Gamma_0(N) \setminus \mathbb{H}^*.$$

The set $X_0(N)$ can be endowed with a complex structure and it is a compact Riemann surface. It is of interest in number theory to embed this Riemann surface into the projective or affine space and find its defining equations. The curve that is an image of such an embedding is called a *model* of the modular curve $X_0(N)$.

The field of rational functions over $\mathbb{C}$ of $X_0(N)$ is generated by j and $j(N\cdot)$. The minimal polynomial of $j(N\cdot)$ over $\mathbb{C}(j)$ is called the classical modular polynomial and it gives the canonical plane model for $X_0(N)$, [14]. However, this polynomial is hard to compute and has enormous coefficients so it is not of practical use. There are few methods for finding different models for the modular curve $X_0(N)$, [5,10,11,15,16]. One method uses the canonical embedding of Riemann surfaces in the projective space, [5]. Using the connection of modular forms on $\Gamma_0(N)$ (or in general on any Fuchsian group of the first kind) with the differentials on $X_0(N)$, G. Muić has searched for models of $X_0(N)$ by constructing maps into the projective space using modular forms of arbitrary weight, [10,11]. We use this method to construct models of $X_0(N)$ into the projective plane $\mathbb{P}^2$, as in [11], using eta-quotients.

There are always two eta-quotients that are modular forms of weight 12 for $\Gamma_0(N)$ for every N: Ramanujan delta function Δ and its rescaling $\Delta_N = \Delta(N\cdot)$. We search for a third function so that the map in the projective plane defined with these three modular forms is a birational equivalence.

In [4] it is proved that the unique normalized modular form of weight 12 for $\Gamma_0(N)$ with maximal order of vanishing at the cusps ∞ is an eta-quotient when the genus of $\Gamma_0(N)$ equals zero. In Sect. 4 we generalize this claim and find those numbers N for which the unique normalized modular form of weight 12 for $\Gamma_0(N)$ whose only zero occurs at infinity is an eta-quotient.

Theorem 1 *The unique normalized modular form of weight* 12 *for* $\Gamma_0(N)$ *which only vanishes at infinity is an eta-quotient if and only if* N *belongs to one of the sets* S_1, S_2, *or* S_3 *defined by*

$$S_1 = \left\{p^n : p \in \{2, 3, 5, 7, 13\}, n \geq 1\right\},$$
$$S_2 = \left\{p_1^{n_1} p_2^{n_2} : p_1 \in \{2, 3, 5\}, p_2 \in \{3, 5, 7, 13\},\ p_1 \neq p_2,\ p_1 p_2 < 40,\ n_1, n_2 \geq 1\right\},$$
$$S_3 = \left\{p_1^{n_1} p_2^{n_2} p_3^{n_3} : p_1 = 2, p_2 \in \{3, 5\}, p_3 \in \{5, 7, 13\},\ p_2 + p_3 < 17, n_1, n_2, n_3 \geq 1, \right\},$$

where p_1, p_2, *and* p_3 *are mutually distinct primes.*

In Sect. 4, in (14),(16),(18) we give the precise form of these eta-quotients which we denote by $\Delta_{N,12}$.

Using functions $\Delta_{N,12}$, Δ, and Δ_N, we construct a map from $X_0(N)$ to the projective plane $\mathbb{P}^2$, namely

$$\mathfrak{a}_z \mapsto (\Delta_{N,12}(z) : \Delta(z) : \Delta(Nz)) \tag{1}$$

and prove that in some cases this map is a birational equivalence. We believe that this map is birational equivalence in all cases and have some numerical results that support this claim but have not proved it yet.

Theorem 2 *Assume one of the following:*

(i) $N \in S_1$ *and* $\Delta_{N,12}$ *is defined as in* (14),
(ii) N *has the form* $2^n 3^m$, $2^n 5^m$, $2^n 13^m$, *or* $3^n 5^m$ *and* $\Delta_{N,12}$ *is defined as in* (16),
(iii) $N = 2^{n_1} 3^{n_2} 7^{n_3}$ *for* $n_1, n_2, n_3 \geq 1$ *and* $\Delta_{N,12}$ *is defined as in* (18).

The modular curve $X_0(N)$ *is birationally equivalent to the curve*

$$\mathcal{C}(\Delta_{N,12}, \Delta, \Delta_N) \subseteq \mathbb{P}^2,$$

which has degree equal to

$$\dim M_{12}(\Gamma_0(N)) + g(\Gamma_0(N)) - 2 = N \prod_{\substack{p|N \\ p \text{ prime}}} (1 + 1/p) - 1.$$

We give a table of some equations for the curves $\mathcal{C}(\Delta_{N,12}, \Delta, \Delta_N)$.

2 Preliminaries

The group $\Gamma_0(N)$ is a modular group, i.e., a subgroup of

$$\Gamma(1) = SL_2(\mathbb{Z})$$

of finite index which is equal to the value of the Dedekind Psi function

$$[\Gamma(1) : \Gamma_0(N)] = N \prod_{p|N} (1 + 1/p) = \Psi(N). \tag{2}$$

The group $\Gamma_0(N)$ has $\sum_{0<d|N} \phi((d, N/d))$ cusps. As a set of representatives of inequivalent cusps we can take the set

$$\mathcal{C}_N = \left\{ \frac{a}{d} \, : \, d|N, (a, d) = 1, a \in (\mathbb{Z}/k\mathbb{Z})^* \text{ for } k = (d, N/d) \right\}. \tag{3}$$

There are $\Phi((d, N/d))$ cusps with denominator d, for each divisor d of N. There are always two cusps for $\Gamma_0(N)$, for every N: one cusps with denominator 1 which we denote $\mathfrak{o}$ and one cusp with denominator N which we denote as cusp ∞.

The Dedekind eta-function is defined by the infinite product

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \quad z \in \mathbb{H}, \quad q = q(z) := e^{2\pi i z}.$$

An *eta-quotient* of level N is a finite product of the form

$$f(z)=\prod_{\delta|N}\eta(\delta z)^{r_\delta},\quad r_\delta\in\mathbb{Z}.$$

Every eta-quotient is a holomorphic function of the upper half plane with no zeroes on the upper half plane.

The most famous example is the Ramanujan delta function,

$$\Delta(z)=\eta(z)^{24}=\sum_{n=1}^{\infty}\tau(n)q^n,$$

which is a cusp form of weight 12 on $\Gamma(1)$. Let us denote

$$\Delta_N(z):=\Delta(Nz).$$

We have

$$\Delta,\Delta_N\in M_{12}(\Gamma_0(N)),\ \text{for all } N,$$

so there are always two eta-quotients of weight 12 on $\Gamma_0(N)$.

Eta-quotients have some beautiful properties. They have integral Fourier expansions at the cusp ∞, using their modular transformation properties, we can calculate their Fourier expansions at other cusps (see [6]).

Ligozat in [7], Proposition 3.2.8 proved the formula for the order of vanishing of an eta-quotient at the cusps with denominator d:

$$\frac{N}{24}\sum_{\delta|N}\frac{(\delta,d)^2r_\delta}{(N,d^2)\delta}. \tag{4}$$

Using modular transformation properties of the Dedekind eta-function, see [1], Theorem 3.4, we can deduce the modular transformation properties of eta-quotients. This result in various forms can be found in [13], Theorem 1.64, [7], Proposition 3.2.1, or [12] Theorem 1.

For a divisor δ of N, we denote by δ' the number $\delta'\delta=N$.

Theorem 3 *If*

(1) $\sum_{\delta|N}\delta r_\delta\equiv 0\pmod{24}$,

(2) $\sum_{\delta|N}\delta' r_\delta\equiv 0\pmod{24}$,

(3) $\prod_{\delta|N}\delta'^{r_\delta}$ *is a square of a rational number,*

then $\prod_{\delta|N}\eta(\delta z)^{r_\delta}$ *is a weakly holomorphic modular form on* $\Gamma_0(N)$ *of weight* $k=\frac{1}{2}\sum_\delta r_\delta$.

If we require that

(4) $\frac{N}{24}\sum_{\delta|N}\frac{(\delta,d)^2 r_\delta}{(N,d^2)\delta}\geq 0$, *for all divisors d of N,*

then $\prod_{\delta|N}\eta(\delta z)^{r_\delta}$ *is a modular form on $\Gamma_0(N)$ of weight* $k=\frac{1}{2}\sum_\delta r_\delta$.

The conditions (4) in Theorem 3 can be written in matrix form. Let

$$A_N=\left(\frac{N(\delta,d)^2}{(N,d^2)\delta}\right)_{d,\delta} \tag{5}$$

with d, δ running over the divisors of N. This is a square matrix of size $\sigma_0(N)$, where $\sigma_0(N)$ is the number of positive divisors of N. This matrix is called the *order matrix*, [3]. If we write the exponents r_δ of an eta-quotient $\prod_{\delta|N}\eta(\delta z)^{r_\delta}$ as a column vector in increasing order

$$\mathbf{r}=(r_\delta)_\delta,$$

then the condition (4) states that the entries of the vector $A_N\mathbf{r}$ are non-negative.

The matrix A_N is invertible over $\mathbb{Q}$ for every N and if the prime factorisation of N is $N=p_1^{n_1}\cdots p_s^{n_s}$, then the matrix A_N is the Kronecker product, see [2], Proposition 1.41,

$$A_N=A_{p_1^{n_1}}\bigotimes\cdots\bigotimes A_{p_s^{n_s}}. \tag{6}$$

3 Maps $X_0(N)\to\mathbb{P}^2$ via modular forms

Let us describe the construction of a map from the modular curve to the projective plane.

Let $k\geq 2$ be an even integer such that $\dim M_k(\Gamma_0(N))\geq 3$. For our purposes, we assume that k is even but it can also be an odd number greater than 3.

Let $f,g,h\in M_k(\Gamma_0(N))$ be three linearly independent modular forms. Let us denote by $\mathfrak{a}_z$ the image of $z\in\mathbb{H}$ under the canonical projection $\mathbb{H}\to X_0(N)$.

We construct the holomorphic map $\varphi:X_0(N)\to\mathbb{P}^2$ which is uniquely determined by

$$\varphi(\mathfrak{a}_z)=(f(z):g(z):h(z)), \tag{7}$$

with $\mathfrak{a}_z$ in the complement of the finite set of $\Gamma_0(N)$-orbits of common zeros of f, g, and h.

Every compact Riemann surface can be observed as a smooth irreducible projective curve over $\mathbb{C}$, and functions g/f and h/f are rational functions on $X_0(N)$. Thus, the map φ is actually a rational map

$$\mathfrak{a}_z\mapsto(1:g(z)/f(z):h(z)/f(z)).$$

Since $X_0(N)$ is smooth, the map is regular and the image is an irreducible curve in $\mathbb{P}^2$. The image is not constant because the functions f, g, and h are linearly independent.

The image of the map (7) is, in most cases, a singular, projective plane curve, which we denote by

$$\mathcal{C}(f, g, h).$$

The field of rational functions of the curve $\mathcal{C}(f, g, h)$, denoted by $\mathbb{C}(\mathcal{C}(f, g, h))$, is isomorphic to a subfield of $\mathbb{C}(X_0(N))$, the field of rational functions of the modular curve $X_0(N)$. By definition, the degree of the map φ, which will be denoted by

$$d(f, g, h),$$

is equal to the degree of the field extension

$$d(f, g, h) = [\mathbb{C}(X_0(N)) : \mathbb{C}(\mathcal{C}(f, g, h))].$$

For a meromorphic function f on a Riemann surface X, we can define its divisor of poles by

$$\mathrm{div}_\infty(f) = \sum_{\substack{a \in X \\ \nu_a(f) < 0}} (-\nu_a(f)) \cdot a,$$

where $\nu_a(f)$ denotes the order of the function f at the point $a \in X$. This is a positive divisor on X. As for every divisor on a compact Riemann surface, we define its degree by

$$\deg(\mathrm{div}_\infty(f)) = \sum_{a \in X} \mathrm{div}_\infty(f)(a).$$

There is a simple criterion for the map φ to be of degree, 1 i.e., to be a birational equivalence. It can be found in [16], Lemma 1 or [11], Lemma 5-2.

Lemma 1 *The degree $d(f, g, h)$ of the map (7) divides the numbers*

$$\deg(div_\infty(g/f)) \quad and \quad \deg(div_\infty(h/f)). \tag{8}$$

A sufficient condition for the map φ to be a birational equivalence is that these numbers are relatively prime,

$$\gcd(\deg(div_\infty(g/f)), \deg(div_\infty(h/f))) = 1. \tag{9}$$

Let us define the divisor of a modular form and its degree.

Let $f \in M_k(\Gamma_0(N))$, $f \neq 0$. For each $\mathfrak{a} \in X_0(N)$, we can define the order of vanishing $\nu_{\mathfrak{a}}(f)$ of f at $\mathfrak{a}$. This number can be rational when $\mathfrak{a}$ is an elliptic point, otherwise it is integral.

We define the divisor of f as

$$\operatorname{div}(f) = \sum_{\mathfrak{a} \in X_0(N)} \nu_{\mathfrak{a}}(f) \cdot \mathfrak{a}.$$

Its degree is

$$\deg(\operatorname{div}(f)) = \sum_{\mathfrak{a} \in X_0(N)} \nu_{\mathfrak{a}}(f).$$

The degree of the divisor of a modular form in $M_k(\Gamma_0(N))$ equals

$$\deg(\operatorname{div}(f)) = k(g(\Gamma_0(N)) - 1) + \frac{k}{2}\left(\nu_\infty(\Gamma_0(N)) + \sum_{\mathfrak{a} \in X_0(N),\ \text{elliptic}} (1 - 1/e_{\mathfrak{a}})\right),$$

where $g(\Gamma_0(N))$ is the genus of $X_0(N)$; $\nu_\infty(\Gamma_0(N))$ is the number of inequivalent cusps; and $e_{\mathfrak{a}}$ is the index of ramification at an elliptic point $\mathfrak{a}$, as can be found in [8], Theorem 2.3.3.

By subtracting the possible non-integer parts of contributions at elliptic points, we obtain an integral divisor D_f attached to the modular form $f \in M_k(\Gamma_0(N))$, defined by

$$D_f = \sum_{\mathfrak{a} \in X_0(N)} [\nu_f(\mathfrak{a})] \cdot \mathfrak{a},$$

where $[x]$ denotes the largest integer $\leq x$, and it has the following degree (see [9], Lemma 4-1 or [8], Thm. 2.5.2)

$$\deg D_f = \dim M_k(\Gamma_0(N)) + g(\Gamma_0(N)) - 1. \tag{10}$$

For more details on divisors of modular forms and attached integral divisors, we refer to [9, Sect. 4] and [8, Sect. 2.3].

The formula for the genus of $X_0(N)$ satisfies (see [8, Theorem 4.2.11])

$$g(\Gamma_0(N)) = 1 + \frac{[\Gamma(1) : \Gamma_0(N)]}{12} - \frac{\nu_2(\Gamma_0(N))}{4} - \frac{\nu_3(\Gamma_0(N))}{3} - \frac{\nu_\infty(\Gamma_0(N))}{2},$$

where $\nu_j(\Gamma_0(N))$ stands for the number of inequivalent elliptic points of order j, for $j = 2, 3$, and the formula for the dimension of $M_{12}(\Gamma_0(N))$ (see [8, Theorem 2.5.2]) is

$$\dim M_{12}(\Gamma_0(N)) = 11(g(\Gamma_0(N)) - 1) + 3\nu_2(\Gamma_0(N)) + 4\nu_3(\Gamma_0(N)) + 6\nu_\infty(\Gamma_0(N)).$$

A simple computation yields the formula

$$\dim M_{12}(\Gamma_0(N)) + g(\Gamma_0(N)) - 1 = [\Gamma(1) : \Gamma_0(N)]\,. \tag{11}$$

The following formula, which we now state for the case of the modular group $\Gamma_0(N)$, is proved in [11, Corollary 3.7].

Theorem 4 *Assume $k \geq 2$ is an even integer such that $\dim M_k(\Gamma_0(N)) \geq 3$. Let $f, g, h \in M_k(\Gamma_0(N))$ be linearly independent and φ the map in (7). Then:*

$$\begin{aligned} d(f,g,h)\deg \mathcal{C}(f,g,h) &= \dim M_k(\Gamma_0(N)) + g(\Gamma_0(N)) - 1 \\ &\quad - \sum_{\mathfrak{a}\in X_0(N)} \min(D_f(\mathfrak{a}), D_g(\mathfrak{a}), D_h(\mathfrak{a})), \end{aligned} \tag{12}$$

where $\deg \mathcal{C}(f,g,h)$ is the degree of the plane curve $\mathcal{C}(f,g,h)$ (the degree of its defining polynomial) and D_f, D_g, and D_h are the integral divisors attached to the modular forms f, g, and h.

If we compute the degree of the curve $\mathcal{C}(f,g,h)$, which can be done by computation with the Fourier coefficients of f, g, and h (a method that can only be performed for small values of N), using the formula (12) we can calculate the degree of the map.

4 Eta-quotients of weight 12 and models of $X_0(N)$

Let us denote by $\Delta_{N,12}$ the unique eta-quotient

$$\Delta_{N,12}(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta},$$

which is a modular form in $M_{12}(\Gamma_0(N))$ and only vanishes at the infinity cusp of $\Gamma_0(N)$.

For which N such function exists? We first will look at three cases, depending on the number of prime factors of N.

In the first case, when $N = p^n$ with p a prime number, we define a set

$$S_1 = \left\{p^n : p \in \{2,3,5,7,13\}\,, n \geq 1\right\}. \tag{13}$$

For $N = p^n \in S_1$, let us look at the following eta-quotient:

$$F_N(z) := \frac{\eta(p^n z)^{pr}}{\eta(p^{n-1}z)^r}, \quad \text{with} \quad r = \frac{24}{p-1}. \tag{14}$$

Now we check that these functions are modular forms and compute their order of vanishing at infinity by checking the conditions of Theorem 3. We have

$$\sum_{\delta|N}\delta r_\delta = -p^{n-1}r + p^n(24+r) = p^{n-1}((p-1)r+24p) = 24(p+1)p^{n-1}.$$

Order of F_N at the cusp ∞ is $1/24\sum_\delta \delta r_\delta = (p+1)p^{n-1}$ and we see that this function has maximal order at the cusp ∞ by the valence theorem since the index of $\Gamma_0(p^n)$ in $\Gamma(1)$ equals $(p+1)p^{n-1}$. The other conditions are also satisfied (orders at all other cusps must be equal to zero):

$$\sum_{\delta|N}\delta' r_\delta = -pr + 24 + r = \frac{24}{p-1}(1-p) + 24 = 0 \text{ (order at the cusp } \mathfrak{o} \text{ equals 0)}$$

$$\prod_{\delta|N}\delta^{r_\delta} = p^{-r(n-1)}p^{n(24-r)} = p^{-2rn+r+24n} \text{ is a square of an integer because } r \text{ is even}$$

$$\sum_{\delta|N} r_\delta = -r + 24 + r = 2\cdot 12.$$

We check the conditions (4) of Theorem 3 to see that this function is a modular form. First, if $d = p^i$ with $i < n$, we have

$$\begin{aligned}\sum_{\delta|N}\frac{(\delta,d)^2 r_\delta}{\delta} &= \frac{(p^n,p^i)^2}{p^n}\frac{24p}{p-1} + \frac{(p^{n-1},p^i)^2}{p^{n-1}}\frac{-24}{p-1}\\ &= \frac{24}{p^{n-1}(p-1)}\cdot(p^{2i}-p^{2i}) = 0,\end{aligned}$$

and in the case $d = p^n$ we have

$$\begin{aligned}\sum_{\delta|N}\frac{(\delta,d)^2 r_\delta}{\delta} &= \frac{(p^n,p^n)^2}{p^n}\frac{24p}{p-1} + \frac{(p^{n-1},p^n)^2}{p^{n-1}}\frac{-24}{p-1}\\ &= \frac{24}{p^{n-1}(p-1)}\cdot(p^{2n}-p^{2(n-1)})\\ &= \frac{24}{p-1}\cdot p^2(p^2-1) \geq 0,\end{aligned}$$

since $p \geq 2$. We conclude that $F_N \in M_{12}(\Gamma_0(p^n))$, and so

$$F_N = \Delta_{N,12}$$

for $N \in S_1$.

In the second case $N = p_1^{n_1} p_2^{n_2}$, i.e., N has two distinct prime factors. Let

$$S_2 = \left\{ p_1^{n_1} p_2^{n_2} : p_1 \in \{2,3,5\}, p_2 \in \{3,5,7,13\}, \right. \\ \left. p_1 \neq p_2,\ p_1 p_2 < 40,\ n_1, n_2 \geq 1 \right\} \tag{15}$$

and

$$G_N(z) := \frac{\eta(p_1^{n_1-1} p_2^{n_2-1} z)^r \eta(p_1^{n_1} p_2^{n_2} z)^{p_1 p_2 r}}{\eta(p_1^{n_1} p_2^{n_2-1} z)^{p_1 r} \eta(p_1^{n_1-1} p_2^{n_2} z)^{p_2 r}} \tag{16}$$

for $N \in S_2$ with

$$r = \frac{24}{(p_1 - 1)(p_2 - 1)}.$$

We check the conditions of the Theorem 3:

$$\begin{aligned}
\sum_{\delta | N} \delta r_\delta &= p_1^{n_1-1} p_2^{n_2-1} r (1 + p_1^2 p_2^2 - p_1^2 - p_2^2) \\
&= p_1^{n_1-1} p_2^{n_2-1} \frac{24}{(p_1 - 1)(p_2 - 1)} (p_1 + 1)(p_1 - 1)(p_2 + 1)(p_2 - 1) \\
&= 24 p_1^{n_1-1} p_2^{n_2-1} (p_1 + 1)(p_2 + 1).
\end{aligned}$$

Order at the cusp ∞ equals $1/24 \sum_\delta \delta r_\delta = p_1^{n_1-1} p_2^{n_2-1} (p_1 + 1)(p_2 + 1)$ which is the maximal order of the zero by the valence theorem since the index of $\Gamma_0(p_1^{n_1} p_2^{n_2})$ is $p_1^{n_1-1} p_2^{n_2-1}(p_1 + 1)(p_2 + 1)$. Order at all other cusps must be equal to zero. The remaining conditions are

$$\begin{aligned}
\sum_{\delta | N} \delta' r_\delta &= p_1 p_2 r + p_1 p_2 r - p_1 p_2 r - p_1 p_2 r = 0, \\
\sum_{\delta | N} r_\delta &= r(1 + p_1 p_2 - p_1 - p_2) = 24, \\
\prod_{\delta | N} \delta'^{r_\delta} &= (p_1 p_2)^r p_2^{-p_1 r} p_1^{-p_2 r} \\
&= p_1^{\frac{-24}{p_1 - 1}} p_2^{\frac{-24}{p_2 - 1}}.
\end{aligned}$$

We have $p_1, p_2 \in \{2, 3, 5, 7, 13\}$, so the numbers $\frac{-24}{p_1-1}$ and $\frac{-24}{p_2-1}$ are even, and we see that $\prod_{\delta|N} \delta'^{r_\delta}$ is a square of a rational number.

We omit the calculations for conditions (4) but they can be easily checked.

Thus we proved that $G_N \in M_{12}(\Gamma_0(p_1^{n_1} p_2^{n_2}))$,

$$G_N = \Delta_{N,12}$$

for $N \in S_2$ and has a zero of maximal order at the cusp ∞.

In the third case, for $N = p_1^{n_1} p_2^{n_2} p_3^{n_3}$ with three distinct prime factors, we define a set

$$S_3 = \left\{ p_1^{n_1} p_2^{n_2} p_3^{n_3} : p_1 = 2, p_2 \in \{3, 5\}, p_3 \in \{5, 7, 13\}, \ p_2 + p_3 < 17, n_1, n_2, n_3 \geq 1, \right\}, \quad (17)$$

where p_1, p_2, and p_3 are mutually distinct prime numbers. Furthermore, we look at the following eta-quotient:

$$H_N(z) = \frac{\eta(p_1^{n_1} p_2^{n_2-1} p_3^{n_3-1} z)^{p_1 r} \eta(p_1^{n_1-1} p_2^{n_2} p_3^{n_3-1} z)^{p_2 r} \eta(p_1^{n_1-1} p_2^{n_2-1} p_3^{n_3} z)^{p_3 r} \eta(p_1^{n_1} p_2^{n_2} p_3^{n_3} z)^{p_1 p_2 p_3 r}}{\eta(p_1^{n_1-1} p_2^{n_2-1} p_3^{n_3-1} z)^{r} \eta(p_1^{n_1} p_2^{n_2} p_3^{n_3-1} z)^{p_1 p_2 r} \eta(p_1^{n_1} p_2^{n_2-1} p_3^{n_3} z)^{p_1 p_3 r} \eta(p_1^{n_1-1} p_2^{n_2} p_3^{n_3} z)^{p_2 p_3 r}} \quad (18)$$

for $N \in S_3$, with

$$r = \frac{24}{(p_1 - 1)(p_2 - 1)(p_3 - 1)}.$$

We check the conditions of Theorem 3.

$$\begin{aligned} \sum_{\delta | N} \delta r_\delta &= p_1^{n_1-1} p_2^{n_2-1} p_3^{n_3-1} r (p_1^2 + p_2^2 + p_3^2 + p_1^2 p_2^2 p_3^2 - 1 - p_1^2 p_2^2 - p_1^2 p_3^2 - p_2^2 p_3^2) \\ &= 24 p_1^{n_1-1} p_2^{n_2-1} p_3^{n_3-1} (p_1 + 1)(p_2 + 1)(p_3 + 1). \end{aligned}$$

Order at the cusp ∞ equals the index of the subgroup $\Gamma_0(p_1^{n_1} p_2^{n_2} p_3^{n_3})$ so this function has zero of maximal order of vanishing at the cusp ∞. Let us check the other conditions:

$$\sum_{\delta | N} \delta' r_\delta = 0,$$

$$\sum_{\delta | N} r_\delta = r(p_1 + p_2 + p_3 + p_1 p_2 p_3 - 1 - p_1 p_2 - p_2 p_3 - p_1 p_3) = 24,$$

$$\begin{aligned} \prod_{\delta | N} \delta'^{r_\delta} &= (p_2 p_3)^{p_1 r} (p_1 p_3)^{p_2 r} (p_1 p_2)^{p_3 r} (p_1 p_2 p_3)^{-r} p_3^{-p_1 p_2 r} p_2^{-p_1 p_3 r} p_1^{-p_2 p_3 r} \\ &= p_1^{\frac{-24}{(p_1-1)}} p_2^{\frac{-24}{(p_2-1)}} p_3^{\frac{-24}{(p_3-1)}}. \end{aligned}$$

We have $p_1, p_2, p_3 \in \{2, 3, 5, 7, 13\}$, so the numbers $\frac{-24}{p_1-1}$, $\frac{-24}{p_2-1}$, and $\frac{24}{(p_3-1)}$ are even. It follows that $\prod_{\delta | N} \delta'^{r_\delta}$ is a square of a rational number. Simple computation shows that H_N also satisfies the conditions (4) of Theorem 3.

We have proved that $H_N \in M_{12}(\Gamma_0(p_1^{n_1} p_2^{n_2} p_3^{n_3}))$ and has maximal order of vanishing at the infinity, $H_N = \Delta_{N,12}$.

Now we prove that these are the only cases when that unique modular form with maximal order of vanishing at infinity is an eta-quotient.

Theorem 5 *The unique normalized modular form $f(z)$ of weight 12 for $\Gamma_0(N)$ which vanishes only at the infinity cusp is an eta-quotient if and only if N belongs to one of the sets S_1, S_2, or S_3 defined above and $f(z)$ is equal to the corresponding function $F_N(z)$, $G_N(z)$, or $H_N(z)$.*

Proof Let $\prod_{\delta|N} \eta(\delta z)^{r_\delta}$ be an eta-quotient of weight 12 for $\Gamma_0(N)$ which only vanishes at infinity. This means that order at all other cusps must be equal to zero, and the order at infinity, whose value is given by the left hand-side of the first expression in Theorem 3 is equal to $[\Gamma(1) : \Gamma_0(N)] = \Psi(N)$, see (2). To determine this eta-quotient is to find the exponents r_δ. To write these conditions in more computable manner, we will write the exponents as a column vector $\mathbf{r} = (r_\delta)_\delta$ and use the order matrix A_N, (5).

We have the following matrix equation:

$$A_N \mathbf{r} = \left(0, \ldots, 0, 24 \cdot p_1^{n_1-1} \cdots p_s^{n_s-1}(p_1+1)\cdots(p_s+1)\right)^\top,$$

i.e., vector $\mathbf{r}$ is the product

$$A_N^{-1} \left(0, \ldots, 0, 24 \cdot p_1^{n_1-1} \cdots p_s^{n_s-1}(p_1+1)\cdots(p_s+1)\right)^\top. \tag{19}$$

For a prime number p, matrix A_{p^n} is a square matrix of size $(n+1)\times(n+1)$, where the element in i-th row and j-th column is given by

$$A_{p^n}(i, j) = p^{n-j+2\min\{i,j\}-\min\{n,2i\}}, \quad 0 \le i, j \le n,$$

and its inverse is given by

$$p^{n-1}(p^2-1)\cdot A_{p^n}^{-1}(i, j) = \begin{cases} p, & \text{if } i = j = 0 \text{ or } i = j = n, \\ -p^{\min\{j,n-j\}}, & \text{if } |i-j| = 1, \\ p^{\min\{j-1,n-j-1\}}(p^2+1), & \text{if } 0 < i = j < n, \\ 0, & \text{otherwise.} \end{cases}$$

Let $N = p_1^{n_1} \ldots p_s^{n_s}$, $p_1 < p_2 < \cdots < p_s$ be the prime factorization of N. The matrix A_N^{-1} is the Kronecker product of inverses of matrices $A_{p_i^{n_i}}$, see (6). The last column of A_N^{-1} has 2^s elements different from 0. First non-zero entry is

$$\frac{(-1)^s 24}{p_1^{n_1-1} \cdots p_s^{n_s-1}(p_1^2-1)\cdots(p_s^2-1)}$$

and from (19) we get the equation

$$r_{p_1^{n_1-1}\cdots p_s^{n_s-1}} = \frac{(-1)^s 24}{(p_1-1)\cdots(p_s-1)}.$$

The condition $r_{p_1^{n_1-1}\cdots p_s^{n_s-1}} \in \mathbb{Z}$ implies $s < 4$.

When $s = 1$, from (19) we have equations

$$\begin{aligned} r_1 &= r_p = \cdots = r_{p^{n-2}} = 0, \\ r_{p^{n-1}} &= \frac{-24}{p-1}, \\ r_{p^n} &= \frac{24p}{p-1}. \end{aligned}$$

The condition $r_{p^{n-1}} \in \mathbb{Z}$ implies $p = 2, 3, 5, 7, 13$, i.e., $N \in S_1$.

For $s = 2$ we have equations

$$\begin{aligned} r_{p^i q^j} &= 0, \text{ for } i < n-1 \text{ or } j < m-1, \\ r_{p^{n-1}q^{m-1}} &= \frac{24}{(p-1)(q-1)} = x, \\ r_{p^{n-1}q^m} &= -qx, \\ r_{p^n q^{m-1}} &= -px, \\ r_{p^n q^m} &= pqx. \end{aligned}$$

The condition $r_{p^{n-1}q^{m-1}} \in \mathbb{Z}$ implies that N must belong to S_2.

For $s = 3$ we have

$$\begin{aligned} & r_{p^i q^j r^k} = 0, \text{ for } i < n-1 \text{ or } j < m-1 \text{ or } k < l-1, \\ & r_{p^{n-1}q^{m-1}r^{l-1}} = \frac{-24}{(p-1)(q-1)(r-1)} = x, \\ & r_{p^n q^{m-1} r^{l-1}} = -px, \quad r_{p^{n-1}q^m r^{l-1}} = -qx, \quad r_{p^{n-1}q^{m-1}r^l} = -rx, \\ & r_{p^{n-1}q^m r^l} = qrx, \quad r_{p^n q^{m-1} r^l} = prx, \quad r_{p^n q^m r^{l-1}} = pqx, \\ & r_{p^n q^m r^l} = -pqrx. \end{aligned}$$

From the condition $r_{p^{n-1}q^{m-1}r^{l-1}} \in \mathbb{Z}$ we have the following possible values

$$(p, q, r) = (2, 3, 5), (2, 3, 7), (2, 3, 13), (2, 5, 7)$$

so N must belong to S_3. The theorem is proved. □

Now we use these functions to construct maps $X_0(N) \to \mathbb{P}^2$. The divisor of the function $\Delta_{N,12}$ from Theorem 5 with respect to the group $\Gamma_0(N)$ is

$$\operatorname{div}(\Delta_{N,12}) = [\Gamma(1) : \Gamma_0(N)]\, \mathfrak{a}_\infty = N \prod_{p|N} (1 + 1/p)\mathfrak{a}_\infty. \tag{20}$$

The divisors of functions Δ, Δ_N with respect to $\Gamma_0(N)$ are given by [11], Lemma 4-2

$$\operatorname{div}(\Delta) = \sum_{c/d \in \mathcal{C}_N} \frac{N}{d(d, N/d)} \mathfrak{a}_{\frac{c}{d}},$$
$$\operatorname{div}(\Delta_N) = \sum_{c/d \in \mathcal{C}_N} \frac{d}{(d, N/d)} \mathfrak{a}_{\frac{c}{d}}. \tag{21}$$

We look at the map $X_0(N) \to \mathbb{P}^2$ given by (1).

$$\mathfrak{a}_z \mapsto (\Delta_{N,12}(z) : \Delta(z) : \Delta(Nz)).$$

Theorem 6 *Assume one of the following:*

(i) $N \in S_1$ *and* $\Delta_{N,12}$ *is the function in* (14),
(ii) N *has the form* $2^n 3^m$, $2^n 5^m$, $2^n 13^m$, *or* $3^n 5^m$ *and* $\Delta_{N,12}$ *is the function in* (16),
(iii) $N = 2^{n_1} 3^{n_2} 7^{n_3}$ *for* $n_1, n_2, n_3 \geq 1$ *and* $\Delta_{N,12}$ *is the function in* (18).

Then the modular curve $X_0(N)$ *is birationally equivalent to the curve*

$$\mathcal{C}(\Delta_{N,12}, \Delta, \Delta_N) \subseteq \mathbb{P}^2,$$

which has degree equal to

$$\dim M_{12}(\Gamma_0(N)) + g(\Gamma_0(N)) - 2 = \Psi(N) - 1.$$

Proof We will look at these three cases separately although the argument is the same. To prove birational equivalence we use Lemma 1.

Case (i): From (20) and (21) we have

$$\deg\left(\operatorname{div}_\infty\left(\frac{\Delta}{\Delta_{p^n,12}}\right)\right) = p^{n-1}(p+1) - 1$$
$$\deg\left(\operatorname{div}_\infty\left(\frac{\Delta_{p^n}}{\Delta_{p^n,12}}\right)\right) = p^{n-1}(p+1-p) = p^{n-1}.$$

Non-trivial divisors of the second number do not divide the first if $n > 1$ so these two numbers are relatively prime for $n > 1$.

From Lemma 1 it follows that the map is a birational equivalence. Since

$$\min(D_\Delta(\mathfrak{a}), D_{\Delta_{N,12}}(\mathfrak{a}), D_{\Delta_N}(\mathfrak{a})) = 0$$

for $\mathfrak{a} \in X_0(N) \setminus \{\infty\}$ and

$$\min(D_\Delta(\infty), D_{\Delta_{N,12}}(\infty), D_{\Delta_N}(\infty)) = 1,$$

from the formula (12) we can calculate the degree of the image curve and it equals $\dim M_{12}(\Gamma_0(N)) + g(\Gamma_0(N)) - 1 - 1$. Formula (11) implies this number equals $p^{n-1}(p+1) - 1$.

Case (ii): From (20) and (21) it follows that

$$\deg\left(\operatorname{div}_\infty\left(\frac{\Delta}{\Delta_{N,12}}\right)\right) = p^{n-1}q^{m-1}(p+1)(q+1)-1$$
$$\deg\left(\operatorname{div}_\infty\left(\frac{\Delta_N}{\Delta_{N,12}}\right)\right) = p^{n-1}q^{m-1}((p+1)(q+1)-pq)$$
$$= p^{n-1}q^{m-1}(p+q+1).$$

These numbers are relatively prime if numbers $(p+q+1)$ and $p^{n-1}q^{m-1}(p+1)(q+1)-1$ are relatively prime. We check all possible cases:

- Let $N=2^n3^m$. Then $2+3+1=6$ and 2 and 3 are prime divisors of 6. But these two numbers do not divide $2^{n-1}\cdot 3^{m-1}\cdot 3\cdot 4-1$.
- Let $N=2^n5^m$. Then $2+5+1=8$ has one prime divisor 2 which does not divide $2^{n-1}5^{m-1}\cdot 3\cdot 6-1$.
- Let $N=2^n13^m$. Then $2+13+1=16$ has one prime divisor 2 which does not divide $2^{n-1}\cdot 13^{m-1}\cdot 3\cdot 14-1$.
- Let $N=3^n5^m$. Then $3+5+1=9$ has one prime divisor 3 which does not divide $3^{n-1}\cdot 5^{m-1}\cdot 4\cdot 6-1$.

From Lemma 1 we have birational equivalence of $X_0(N)$ and $\mathcal{C}(\Delta_{N,12},\Delta,\Delta_N)$.

Case (iii): From (20) and (21) we have

$$\deg\left(\operatorname{div}_\infty\left(\frac{\Delta}{\Delta_{N,12}}\right)\right) = 2^{n_1-1}3^{n_2-1}7^{n_3-1}\cdot 3\cdot 4\cdot 8-1,$$
$$\deg\left(\operatorname{div}_\infty\left(\frac{\Delta_N}{\Delta_{N,12}}\right)\right) = 2^{n_1-1}3^{n_2-1}7^{n_3-1}(3\cdot 4\cdot 8-2\cdot 3\cdot 7),$$
$$= 2^{n_1-1}3^{n_2-1}7^{n_3-1}\cdot 54.$$

Prime divisors of $\deg\left(\operatorname{div}_\infty\left(\frac{\Delta_N}{\Delta_{N,12}}\right)\right)$ belong to the set $\{2,3,7\}$. None of the numbers from this set divides the number $\deg\left(\operatorname{div}_\infty\left(\frac{\Delta}{\Delta_{N,12}}\right)\right)$. These numbers are relatively prime, so by Lemma 1 we conclude that the map defined by functions $\Delta_{N,12}$, Δ, and Δ_N is birational equivalence. The theorem is proved. □

It is our conjecture that the map

$$\mathfrak{a}_z \mapsto (\Delta_{N,12}(z) : \Delta(z) : \Delta_N(z))$$

is birational equivalence for all functions $\Delta_{N,12}$ from Theorem 5 but the argument with divisors of poles is satisfied only in the mentioned cases. In other cases, divisors of poles of functions used are not relatively prime.

As an example, for $N=2^3 7^1=56$ we have

$$\deg\left(\operatorname{div}_\infty\left(\frac{\Delta}{\Delta_{56,12}}\right)\right) = 95,$$

Table 1 Equations for curves $\mathcal{C}(\Delta_{N,12}, \Delta, \Delta_N)$ from Theorem 6

$N = 2$	$x_0x_1 - x_2^2$
$N = 3$	$x_0^2x_1 - x_2^3$
$N = 4$	$x_0^3x_1^2 + 4096x_0^3x_1x_2 + 48x_0^2x_1x_2^2 - x_2^5$
$N = 5$	$x_0^4x_1 - x_2^5$
$N = 7$	$x_0^6x_1 - x_2^7$
$N = 9$	$x_0^8x_1^3 + 531441x_0^8x_1^2x_2 + 282429536481x_0^8x_1x_2^2 + 27894275208x_0^7x_1x_2^3$ $-756x_0^6x_1^2x_2^3 + 975725676x_0^6x_1x_2^4 + 14171760x_0^5x_1x_2^5$ $+74358x_0^4x_1x_2^6 + 72x_0^3x_1x_2^7 - x_2^{11}$
$N = 13$	$x_0^{12}x_1 - x_2^{13}$

$$\deg\left(\mathrm{div}_\infty\left(\frac{\Delta_{56}}{\Delta_{56,12}}\right)\right) = 40.$$

However, we have developed an algorithm that calculates the degree of the resulting curve and with the aid of formula (12) we calculated that in this case the degree of the map equals 1.

At the end, we present some equations for the curves $\mathcal{C}(\Delta_{N,12}, \Delta, \Delta_N)$ in Table 1:

References

1. Apostol, T.M.: Modular Functions and Dirichlet Series in Number Theory. Springer, New York (1990)
2. Bhattacharya, S.: Factorization of holomorphic eta quotients, Ph D thesis, Bonn (2014)
3. Bhattacharya, S.: Finiteness of simple holomorphic eta quotients of a given weight. arXiv:1602.02825 (2016)
4. Choi, D.: Spaces of modular forms generated by eta-quotients. Ramanujan J. **14**, 69–77 (2007)
5. Galbraigth, S.: Equations for modular curves, Ph.D. thesis, Oxford (1996)
6. Köhler, G.: Eta Products and Theta Series Identities. Springer, Berlin (2011)
7. Ligozat, G.: Courbes modulaires de genre 1. Bull. Soc. Math. France [Memoire 43]: 1–80 (1972)
8. Miyake, T.: Modular Forms. Springer, Berlin (2006)
9. Muić, G.: Modular curves and bases for the spaces of cuspidal modular forms. Ramanujan J. **27**, 181–208 (2012)
10. Muić, G.: On embeddings of curves in projective spaces. Monatsh. Math. **173**(2), 239–256 (2014)
11. Muić, G.: On degrees and birationality of the maps $X_0(N) \to \mathbb{P}^2$ constructed via modular forms. Monatsh. Math. **180**(3), 607–629 (2016)
12. Newman, M.: Construction and applications of a certain class of modular functions. Proc. Lond. Math. Soc **3**(7), 334–350 (1956)
13. Ono, K.: The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-Series. In: CBMS Regional Conference Series in Mathematics, vol. 102, Published for the Conference Board of the Mathematical Sciences, Washington, DC (2004)
14. Shimura, G.: Introduction to the arithmetic theory of automorphic functions. Kan Memorial Lectures, No. 1. Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo; Princeton University Press, Princeton, NJ (1971)
15. Shimura, M.: Defining equations of modular curves $X_0(N)$. Tokyo J. Math. **18**(2), 443–456 (1995)
16. Yifan, Y.: Defining equations of modular curves. Adv. Math. **204**, 481–508 (2006)